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Coherent states with $SU(2)$ and $SU(3)$ charges

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Abstract
We define coherent states carrying $SU(2)$ charge by exploiting the Schwinger boson representation of the $SU(2)$ Lie algebra. These coherent states satisfy the continuity property and provide resolution of identity on S^3. We further generalize these techniques to construct the corresponding $SU(3)$ charge coherent states. The $SU(N)$ extension is also discussed.

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1. Introduction

The concept of coherent states was introduced by Schrödinger [1] in the context of a harmonic oscillator. These harmonic oscillator coherent states, also called canonical coherent states, have been useful and studied extensively in physics [2]. The next most important coherent states are spin coherent states or $SU(2)$ coherent states which are associated with angular momentum or the $SU(2)$ group. Like canonical coherent states, they too have found wide applications in different branches of physics such as quantum optics, statistical mechanics, nuclear physics and condensed matter physics [2]. It is known that these spin coherent states can also be constructed using harmonic oscillators by exploiting either the Holstein–Primakov or the Schwinger boson representation of the $SU(2)$ Lie algebra [3–5]. This harmonic oscillator formulation of spin coherent states is appealing because it is simple and analogous to canonical coherent state construction. Further, it bypasses the action of group elements [6] to get the spin coherent states. Motivated by the resulting simplifications, we recently generalized this harmonic oscillator formulation of coherent states to $SU(N)$ group [7]. In this work, we further exploit the above ideas to construct $SU(2)$ and $SU(3)$ charge coherent states defined on S^3 and S^5, respectively. The coherent states carrying $SU(2)$ and $SU(3)$ (non-Abelian) charges in two- and three-mode Fock spaces have been discussed in the past [5, 8–10]. However, they are defined on full complex planes and are different from the $SU(2)$ and $SU(3)$ charge coherent states discussed in this paper which are defined on the compact manifolds S^3 and S^5, respectively. We will further elaborate on these differences as we proceed (section 2.1 and section 3).

The plan of the paper is as follows. We start with a brief description of harmonic oscillator coherent states. The coherent states discussed later will have their roots in this simple construction. In section 2, using two harmonic oscillators, we exploit Schwinger boson representation to construct $SU(2)$ coherent states. This construction is known and is contained in [3]. However, we have included this section to make the presentation self-contained. In section 3, we define $SU(2)$ charge coherent states which satisfy resolution of identity over the $SU(2)$ group manifold S^3. In section 4, we generalize these ideas to $SU(3)$ group. In section 5, we give $SU(N)$ construction.

The harmonic oscillator coherent states are defined as

$$|z\rangle = \sum_{n=0}^{\infty} \frac{(za^\dagger)^n}{n!}|0\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle. \tag{1}$$

These coherent states are associated with the Heisenberg–Weyl group whose Lie algebra is given by

$$[a, a^\dagger] = \mathcal{I}, \qquad [a, \mathcal{I}] = 0, \qquad [a^\dagger, \mathcal{I}] = 0. \tag{2}$$

In (2), $\mathcal{I}$ is the identity operator. The manifold corresponding to the Heisenberg–Weyl group is the complex z plane. The coherent states in (1) are analytic over this group manifold $\mathcal{M}$ and satisfy the resolution of identity:

$$\int_{\mathcal{M}} \mathrm{d}\mu(z) |z\rangle_{\infty\infty}\langle z| \equiv \mathcal{I}. \tag{3}$$

In (3), $\mathrm{d}\mu(z) \equiv \exp(-|z|^2)\,\mathrm{d}z\,\mathrm{d}\bar{z}$ is the measure over $\mathcal{M}$. We now give the construction of $SU(2)$ coherent state which is similar to (1).

2. $SU(2)$ coherent states

The $SU(2)$ group involves three angular momentum generators, J_1, J_2 and J_3, and the Lie algebra is

$$[J^a, J^b] = \mathrm{i}\epsilon_{abc} J^c, \qquad a, b, c = 1, 2, 3. \tag{4}$$

The $SU(2)$ Casimir operator is $\vec{J}\cdot\vec{J}$ with eigenvalue $j(j+1)$, where j is integer or half-integer spin. The angular momentum algebra in (4) can be realized in terms of a doublet of harmonic oscillator annihilation creation operators $\vec{a} \equiv (a_1, a_2)$ and $\vec{a}^\dagger \equiv \left(a_1^\dagger, a_2^\dagger\right)$, respectively [3]. The number operators are $\hat{N}_1 \equiv a_1^\dagger a_1$ and $\hat{N}_2 \equiv a_2^\dagger a_2$. They satisfy the bosonic commutation relations

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}, \qquad i, j = 1, 2. \tag{5}$$

The vacuum state is $|0, 0\rangle$ and the number operator basis is written as $|n_1, n_2\rangle$. It satisfies

$$\hat{N}_1|n_1, n_2\rangle = n_1|n_1, n_2\rangle, \qquad \hat{N}_2|n_1, n_2\rangle = n_2|n_1, n_2\rangle. \tag{6}$$

We can now define the angular momentum operators in (4) as

$$J^a \equiv \tfrac{1}{2} a_i^\dagger (\sigma^a)_{ij} a_j, \tag{7}$$

where σ^a denote the Pauli matrices. It is easy to check that the operators in (7) satisfy (4). Further, as they involve one creation and one annihilation operator, the Casimir ($\mathcal{C}$) is

$$\mathcal{C} = \hat{N}_1 + \hat{N}_2. \tag{8}$$

One can also explicitly check that $\vec{J}\cdot\vec{J} \equiv \frac{1}{4}\mathcal{C}(\mathcal{C}+2) = \frac{1}{4}\vec{a}^{\dagger}\cdot\vec{a}(\vec{a}^{\dagger}\cdot\vec{a}+2)$. Thus, the representations of $SU(2)$ can be characterized by the eigenvalues of the total occupation number operator and the spin value j is equal to $n/2 \equiv (n_1+n_2)/2$. With the $SU(2)$ Schwinger representation (7), we can directly generalize (1) and write down the spin coherent states as[1]

$$
\begin{aligned}
|\vec{z}\rangle_n &= \sqrt{n!}\sum_{n_1,n_2=0}^{n}{}' \frac{z_1^{n_1} z_2^{n_2}}{\sqrt{n_1!n_2!}}|n_1, n_2\rangle \\
&= \sqrt{n!}\sum_{m=0}^{n} \frac{z_1^{n-m} z_2^{m}}{\sqrt{(n-m)!m!}}|n-m, m\rangle.
\end{aligned} \tag{9}
$$

In (9), the prime over the summation sign implies $n_1+n_2 = n = 2j$. It is easy to see that the states $|n_1, n_2\rangle$ with the above constraint form $(2j+1)$-dimensional representation of $SU(2)$ group. Further, (z_1, z_2) is a doublet of complex numbers satisfying the constraint

$$
|z_1|^2 + |z_2|^2 = 1. \tag{10}
$$

Thus, the coherent states in (9) are defined over the sphere S^3. It is easy to check that

$$
\int_{S^3} \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2 \delta(|z_1|^2 + |z_2|^2 - 1)|\vec{z}\rangle_{nn}\langle\vec{z}| = \mathcal{I}_n. \tag{11}
$$

In (11), $\mathcal{I}_n$ is a $(n+1)\times(n+1)$-dimensional unit matrix. By construction, the coherent states in (9) satisfy

$$
\mathcal{C}|\vec{z}\rangle_n = n|\vec{z}\rangle_n. \tag{12}
$$

Thus, the $SU(2)$ coherent states are smoothly defined over the $SU(2)$ group manifold ($\mathcal{M} = S^3$) and are eigenstates of total angular momentum operator $\left(J^2 = J_1^2 + J_2^2 + J_3^2\right)$. In terms of angular momentum basis, $|j; m\rangle \equiv |n_1 = j+m, n_2 = j-m\rangle$, the spin coherent states can be written as

$$
|\vec{z}\rangle_{n=2j} = \sqrt{2j!}\sum_{m=-j}^{j} \frac{1}{\sqrt{(j+m)!(j-m)!}} z_1^{j+m} z_2^{j-m}|j; m\rangle. \tag{13}
$$

We can also obtain (13) by directly operating the $SU(2)$ group element $U(\theta, \phi, \psi) \equiv \exp(\mathrm{i}\phi J_3)\exp(\mathrm{i}\theta J_2)\exp(\mathrm{i}\psi J_3)$ on the highest weight state in the jth representation [6]:

$$
\begin{aligned}
|\theta, \phi, \psi\rangle\rangle_j &= U(\theta, \phi, \psi)|j; j\rangle, \\
&= \exp\mathrm{i}(j\psi)\sum_{m=-j}^{+j} C_m(\theta, \phi)|j; m\rangle.
\end{aligned} \tag{14}
$$

In (14),

$$
C_m(\theta, \phi) = \sqrt{\frac{2j!}{(j+m)!(j-m)!}}\exp(\mathrm{i}m\phi)\left(\sin\frac{\theta}{2}\right)^{j-m}\left(\cos\frac{\theta}{2}\right)^{j+m}. \tag{15}
$$

The identification

$$
z_1 \equiv \mathrm{e}^{\mathrm{i}\frac{\psi}{2}}\,\mathrm{e}^{\mathrm{i}\frac{\phi}{2}}\cos\frac{\theta}{2}, \qquad z_2 \equiv \mathrm{e}^{\mathrm{i}\frac{\psi}{2}}\,\mathrm{e}^{-\mathrm{i}\frac{\phi}{2}}\sin\frac{\theta}{2} \tag{16}
$$

shows the equivalence of the harmonic oscillator (9) and group action (14) constructions. We now exploit the Schwinger boson representation (7) to construct new types of coherent states which cannot be generated by simple group action.

[1] In [4], the $SU(2)$ coherent state construction is through the Holstein–Primakov representation of the $SU(2)$ Lie algebra.

2.1. $SU(2)$ charge coherent states

So far, the $SU(2)$ coherent states were defined with fixed angular momentum j and they were linear combinations of the states $|j;m\rangle$ with m varying from $-j$ to $+j$. The corresponding weight factors were definite analytic functions on $SU(2)$ manifold. We now define new types of $SU(2)$ coherent states which carry fixed charge (m) and are linear combinations of the states $|j;m\rangle$ with j varying from $|m|$ to ∞. Again, the corresponding weight factors are certain analytic functions on $SU(2)$ manifold. For convenience, we define the $SU(2)$ charge operator $\mathcal{Q}$ to be twice the third component of the angular momentum, i.e.,

$$\mathcal{Q} = a_1^\dagger a_1 - a_2^\dagger a_2. \tag{17}$$

The $SU(2)$ fixed charge ($q = 2m$) coherent states are defined as

$$\begin{aligned}|\vec{z}\rangle_q \equiv |z_1, z_2\rangle_q &= \sum_{n_1,n_2=0}^{\infty}{}' \sqrt{\frac{(n_1+n_2+1)!}{n_1!n_2!}} z_1^{n_1} z_2^{n_2} |n_1, n_2\rangle \\ &= \sum_{r=0}^{\infty} \sqrt{\frac{(q+2r+1)!}{(q+r)!r!}} z_1^{(q+r)} z_2^{r} |q+r, r\rangle.\end{aligned} \tag{18}$$

In (18), the prime over $\sum$ implies that $n_1 - n_2 = q$. Therefore, the fixed charge coherent states in (18) satisfy

$$\mathcal{Q}|\vec{z}\rangle_q = q|\vec{z}\rangle_q. \tag{19}$$

Further, as the total number operator $\mathcal{C}$ and a_1a_2 commute with $\mathcal{Q}$, it is easy to check that

$$f(\mathcal{C}) a_1 a_2 |\vec{z}\rangle_q = z_1 z_2 |\vec{z}\rangle_q, \tag{20}$$

where $f(\mathcal{C}) \equiv \frac{1}{\sqrt{(\mathcal{C}+3)(\mathcal{C}+2)}}$. We note that in the context of canonical coherent states, the states satisfying $f(a^\dagger a)a|z\rangle = z|z\rangle$ are known as non-linear coherent states and have been extensively studied (see [9] and references therein).

The charge coherent states satisfy resolution of identity over the $SU(2)$ group manifold S^3 (see the appendix):

$$\int_{S^3} \mathrm{d}^2 z_1\, \mathrm{d}^2 z_2 \delta(|z_1|^2 + |z_2|^2 - 1) |\vec{z}\rangle_{q\,q}\langle\vec{z}| = \mathcal{I}_q. \tag{21}$$

In (21), $\mathcal{I}_q$ is the infinite-dimensional unit matrix. Thus, we have constructed the charge coherent states on S^3. Therefore, using resolution of identity (11), we can express them in terms of fixed angular momentum coherent states (9). The expansion is

$$|z_1, z_2\rangle_q = \sum_{n=0}^{\infty} \int_{S^3} \mathrm{d}^2 w_1\, \mathrm{d}^2 w_{2\,n}\langle w_1, w_2 | z_1, z_2\rangle_q |w_1, w_2\rangle_n. \tag{22}$$

Putting $n = 2j$ in (9) and $q = 2j_3$ in (18), it is easy to see that $|\vec{z}\rangle_{q=2j_3}$ has non-zero overlap with $|\vec{z}\rangle_{n=2j}$ iff $j \geqslant |j_3|$. More precisely,

$$\begin{aligned}{}_n\langle w_1, w_2 | z_1, z_2\rangle_q &= \frac{\sqrt{n!(n+1)!}}{(p+q)!p!} (\bar{w}_1 z_1)^{p+q} (\bar{w}_2 z_2)^p, \qquad \text{if} \quad n = 2p+q, \\ &= 0, \qquad \text{otherwise}.\end{aligned} \tag{23}$$

In (23), p is a non-negative integer such that n is positive. It is interesting to write (18) in the angular momentum basis $|j;m\rangle$,

$$|\vec{z}\rangle_{q=2m} \equiv \sum_{j=|m|}^{\infty} \sqrt{\frac{(2j+1)!}{(j+m)!(j-m)!}} z_1^{(j+m)} z_2^{j-m} |j;m\rangle. \tag{24}$$

Note that the fixed charge coherent state (24), unlike spin coherent states (13), cannot be obtained by a simple group action like in (14) and they are not eigenstates of J^2.

At this stage, it is interesting to compare our formulation of $SU(2)$ charge coherent states with the already existing formulations [5, 8]. In [8], the $SU(2)$ charge coherent states are defined as

$$|\zeta\rangle_q = N_q \sum_{n=0}^{\infty} \frac{\zeta^n}{\sqrt{[n!(n+q)!]}} |n+q, n\rangle. \tag{25}$$

In (25), N_q is the normalization factor and ζ are the coordinates of the complex plane. Further, the above coherent states satisfy resolution of identity:

$$\int_{R^2} \frac{\mathrm{d}^2\zeta}{\pi} \phi_q(\zeta) |\zeta\rangle_{q\,q}\langle\zeta| = \mathcal{I}_q. \tag{26}$$

In (26), $\phi_q(\zeta) = J_q(2\mathrm{i}|\zeta|)K_q(2|\zeta|)$, where J_q and K_q are the Bessel and modified Bessel functions, respectively. In [5], $SU(2)$ fixed charge coherent states $|z\rangle_{j,q}$ are constructed on the complex plane z which are eigenstates of J^2 as well as the charge operator J^3. This has been possible because of the use of three harmonic oscillators (a_1, a_2, a_3) to construct the $SU(2)$ Lie algebra:

$$\hat{J}^i = -\mathrm{i}\epsilon^{ijk} a_j^\dagger a_k. \tag{27}$$

Denoting $|n_1, n_2, n_3\rangle = (n_1!n_2!n_3!)^{-\frac{1}{2}} \left(a_1^\dagger\right)^{n_1} \left(a_2^\dagger\right)^{n_2} \left(a_3^\dagger\right)^{n_3} |0, 0, 0\rangle$, the $SU(2)$ charged coherent states are defined as

$$|\zeta\rangle_{j,q} = \sum_{n=0}^{\infty} \left(\frac{2^j(n+j)}{(2n+2j+1)!n!} \right)^{\frac{1}{2}} \zeta^n |j, q, j+2n\rangle. \tag{28}$$

They satisfy [5]

$$J \cdot J|\zeta\rangle_{j,q} = j(j+1)|\zeta\rangle_{j,q}, \qquad \mathcal{Q}|\zeta\rangle_{j,q} = q|\zeta\rangle_{j,q}, \qquad a \cdot a|\zeta\rangle_{j,q} = \zeta|\zeta\rangle_{j,q}, \tag{29}$$

where $a \cdot a = a_1a_1 + a_2a_2 + a_3a_3$. The resolution of identity is given by

$$\sum_{j=0}^{\infty} \sum_{q=-j}^{j} \int_{R^2} \frac{\mathrm{d}^2\zeta}{2\pi} \Phi_j(|\zeta|) |\zeta\rangle_{j,q\,j,q}\langle\zeta| = \mathcal{I}. \tag{30}$$

In (30), $\Phi_j(|\zeta|)$ are related to modified Bessel functions [5]. Thus, the charged coherent states $|z\rangle_q$ defined in (18) or (24) are different from the ones discussed earlier in [5, 8]. In particular, unlike $|z\rangle_q$ which are defined on the $SU(2)$ group manifold S^3, $|\zeta\rangle_q$ in (25) and $|\zeta\rangle_{j,q}$ in (28) are defined on non-compact manifold R^2. In the next section, we will instead use the three oscillators to define $SU(3)$ coherent states.

3. $SU(3)$ charge coherent states

We will now generalize the fixed $SU(2)$ charge coherent state ideas to the group $SU(3)$. For the sake of simplicity, we will restrict ourselves to $SU(3)$ representations which are completely symmetric. We, therefore, define a triplet of harmonic oscillator creation and annihilation operators satisfying

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}, \qquad i, j = 1, 2, 3. \tag{31}$$

Let $\frac{\lambda^a}{2}, a = 1, 2, \ldots, 8$ be the generators of $SU(3)$ in the fundamental representation; they satisfy the $SU(3)$ Lie algebra $\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = \mathrm{i} f^{abc}\frac{\lambda^c}{2}$. Let us define the following operators [11]:

$$Q^a = \tfrac{1}{2} a_i^\dagger \lambda^a_{ij} a_j. \tag{32}$$

More explicitly[2],

$$\begin{aligned} Q^3 &= \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right), & Q^8 &= \frac{1}{2\sqrt{3}}\left(a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3\right),\\ Q^1 &= \frac{1}{2}\left(a_1^\dagger a_2 + a_2^\dagger a_1\right), & Q^2 &= -\frac{\mathrm{i}}{2}\left(a_1^\dagger a_2 - a_2^\dagger a_1\right),\\ Q^4 &= \frac{1}{2}\left(a_1^\dagger a_3 + a_3^\dagger a_1\right), & Q^5 &= -\frac{\mathrm{i}}{2}\left(a_1^\dagger a_3 - a_3^\dagger a_1\right),\\ Q^6 &= \frac{1}{2}\left(a_2^\dagger a_3 + a_3^\dagger a_2\right), & Q^7 &= -\frac{\mathrm{i}}{2}\left(a_2^\dagger a_3 - a_3^\dagger a_2\right). \end{aligned} \tag{33}$$

It is clear that the total number operator $\mathcal{C} = a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3$ commutes with all the $SU(3)$ generators in (33).

The $SU(3)$ coherent states analogous to (9) are

$$|\vec{z}\rangle_n \equiv |z_1, z_2, z_3\rangle_n = \sqrt{n!} \sum_{n_1,n_2,n_3=0}^{n}{}' \frac{z_1^{n_1} z_2^{n_2} z_3^{n_3}}{\sqrt{n_1! n_2! n_3!}} |n_1, n_2, n_3\rangle. \tag{34}$$

In (34), the prime over the summation sign implies $n_1 + n_2 + n_3 = n$. With this constraint, the states $|n_1, n_2, n_3\rangle$ form all the symmetric representations of $SU(3)$. They are of dimensions $\frac{(n+1)(n+2)}{2}$ and the coherent states $|\vec{z}\rangle_n$ satisfy

$$\mathcal{C}|\vec{z}\rangle_n = n|\vec{z}\rangle_n. \tag{35}$$

Further in (34), (z_1, z_2, z_3) is a triplet of complex numbers satisfying the constraint

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1. \tag{36}$$

Note that the $SU(3)$ coherent states in a mixed representation of $SU(3)$ can be defined by introducing a second independent set of oscillators $(b_1^\dagger, b_2^\dagger, b_3^\dagger)$ and defining [7]

$$Q^a = \tfrac{1}{2} a_i^\dagger \lambda^a_{ij} a_j - \tfrac{1}{2} b_i^\dagger \lambda^a_{ji} b_j, \tag{37}$$

leading to a second Casimir $\mathcal{C}' = b^\dagger \cdot b = b_1^\dagger b_1 + b_2^\dagger b_2 + b_3^\dagger b_3$. However, in this work, to keep the discussion simple and analogous to the $SU(2)$ group, which has only symmetric representations, we will be interested only in the symmetric representations of $SU(3)$ with a single Casimir given in (35) and $\mathcal{C}' = 0$. Now the $SU(3)$ construction in (34), (35) and the constraint (36) are analogous to the corresponding $SU(2)$ construction in (9), (12) and the constraint (10), respectively. Further, as in the $SU(2)$ case, it is easy to check that

$$\int \mathrm{d}^2 z_1 \int \mathrm{d}^2 z_2 \int \mathrm{d}^2 z_3 \delta(|z_1|^2 + |z_2|^2 + |z_3|^2 - 1) |\vec{z}\rangle_{nn}\langle \vec{z}| = \mathcal{I}_n. \tag{38}$$

In (38), $\mathcal{I}_n$ is $\frac{1}{2}(n+1)(n+2)$-dimensional unit matrix. Thus, the coherent states in (34) are defined over the sphere S^5. We now define $SU(3)$ charge and hyper-charge operators for the symmetric representations to be

$$\begin{aligned} \mathcal{Q}_1 &\equiv 2Q^3 = \left(a_1^\dagger a_1 - a_2^\dagger a_2\right),\\ \mathcal{Q}_2 &\equiv 2\sqrt{3} Q^8 = a_1^\dagger a_1 + a_2^\dagger a_2 - 2a_3^\dagger a_3. \end{aligned} \tag{39}$$

[2] Note that in [5], $J^1 \equiv 2Q^7$, $J^2 \equiv -2Q^5$ and $J^3 \equiv 2Q^2$ are used to define coherent states with fixed J^3 and $\vec{J} \cdot \vec{J}$.

The $SU(3)$ charge and hyper-charge coherent states are given by

$$|z_1, z_2, z_3\rangle_{q,l} = \sum_{p=0}^{\infty} \sqrt{\frac{(3p+2l-q+2)!}{(p+l)!(p+l-q)!p!}} {z_1}^{p+l} {z_2}^{p+l-q} {z_3}^{p} |p+l, p+l-q, p\rangle. \tag{40}$$

They satisfy

$$\mathcal{Q}_1|z_1, z_2, z_3\rangle_{q,l} = q|z_1, z_2, z_3\rangle_{q,l}, \qquad \mathcal{Q}_2|z_1, z_2, z_3\rangle_{q,l} = (2l-q)|z_1, z_2, z_3\rangle_{q,l}, \tag{41}$$

where $q = q_1$ and $2l = q_1 + q_2$. The coherent states in (40) are generalization of the corresponding $SU(2)$ coherent states in (18) and satisfy resolution of identity property (see the appendix). We can also expand (40) in terms of (34):

$$|z_1, z_2, z_3\rangle_{q,l} = \sum_{n=0}^{\infty} \int_{S^5} \mathrm{d}^3 w_1\, \mathrm{d}^3 w_2\, \mathrm{d}^3 w_3\, {}_n\langle w_1, w_2, w_3|z_1, z_2 z_3\rangle_{q,l} |w_1, w_2, w_3\rangle_n. \tag{42}$$

The overlap is given by

$$\begin{aligned} {}_n\langle\vec{w}|\vec{z}\rangle_{q,l} &= \frac{\sqrt{n!(n+2)!}}{(p+l)!(p+l-q)!p!} (\bar{w}_1 z_1)^{p+l} (\bar{w}_2 z_2)^{p+l-q} (\bar{w}_3 z_3)^{p}, \qquad \text{if} \quad n = 3p+2l-q \\ &= 0, \qquad \text{otherwise}. \end{aligned} \tag{43}$$

In (43), p is any non-negative integer such that n is positive. The above expression for the overlap is analogous to (23) in the case of $SU(2)$.

We now compare our $SU(3)$ charge coherent state construction with the ones given in [9, 10]. These coherent states are defined as[3]

$$|\zeta\rangle_{\bar{q},y} = N_{\bar{q},y} \sum_{m=0}^{\infty} \frac{\zeta^m}{[m!(m+y+\bar{q})!(m+2y-\bar{q})!]^{\frac{1}{2}}} |m+y+\bar{q}, m+2y-\bar{q}, m\rangle. \tag{44}$$

Again, as in the $SU(2)$ case, the $SU(3)$ charge coherent states $|\zeta\rangle_{\bar{q},y}$ are defined over R^2 and thus different from the ones in (40) defined on the compact manifold S^5.

4. $SU(N)$ charge coherent states

We now briefly discuss $SU(N)$ construction by using an N-plet of harmonic oscillators. The coherent states analogous to (9) and (34) are

$$|z_1, z_2, \ldots, z_N\rangle_n = \sqrt{n!} \sum_{n_1,n_2,\ldots,n_N=0}^{n}{}' \frac{z_1^{n_1} z_2^{n_2} \cdots z_N^{n_N}}{\sqrt{n_1! n_2! \cdots n_N!}} |n_1, n_2, \ldots, n_N\rangle. \tag{45}$$

In (45), $n_1 + n_2 + \cdots + n_N = n$ and

$$|z_1|^2 + |z_2|^2 + \cdots + |z_N|^2 = 1. \tag{46}$$

Thus, the coherent states in (45) are defined on S^{2N-1}. Note that, like in the $SU(3)$ case, we have again restricted ourselves to only symmetric representations of $SU(N)$. Now within the symmetric representations, the $(N-1)$ charge operators can be chosen as $\mathcal{Q}_l = a_l^\dagger a_l - a_{l+1}^\dagger a_{l+1}$, $l = 1, 2, \ldots, (N-1)$ and the corresponding eigenvalues will be denoted by q_l. Now we can easily generalize the charge coherent states in (18) and (40) to $SU(N)$ group:

$$|z_1, z_2, \ldots, z_N\rangle_{q_1,q_2,\ldots,q_{N-1}} = \sum_{n_N=0}^{\infty} L \frac{z_1^{n_1} z_2^{n_2} \cdots z_N^{n_N}}{\sqrt{n_1! n_2! \cdots n_N!}} |n_1, n_2, \ldots, n_N\rangle, \tag{47}$$

[3] We are following the notations of [9]. The charges in (40) and (44) are related by $q = 2\bar{q} - y$ and $l = \bar{q} + y$.

where

$$L = \sqrt{(n_1 + n_2 + \cdots + n_N + (N-1))!} \tag{48}$$

and the N occupation numbers $n_i = n_N + \sum_{j=i}^{(N-1)} q_j$ and $i = 1, 2, \ldots, (N-1)$. Like (45), the charge coherent states in (47) satisfy resolution of identity on S^{2N-1}.

5. Summary and discussion

We have constructed $SU(2)$ charge coherent states which are defined and satisfy resolution of identity over the $SU(2)$ group manifold S^3. Further, they are eigenstates of J^3 *and not of* $\vec{J}\cdot\vec{J}$. We then defined $SU(3)$ and $SU(N)$ charge coherent states on S^5 and S^{2N-1}, respectively. The spin coherent states have been extensively used to study the partition functions of $SU(2)$ spin models leading to useful semiclassical descriptions [12]. The $SU(2)$ spin model Hamiltonian is given by $H = \sum_{\langle i,j\rangle} \vec{J}(i)\cdot\vec{J}(j)$, where i and j are site indices and $\langle i, j\rangle$ denotes the nearest neighbours. This Hamiltonian commutes with $\vec{J}(i)\cdot\vec{J}(i), \forall\, i$. Therefore, the fixed angular momentum coherent state basis in (9) has been exploited for the path integral formulation of the partition function [12]. Instead, let us consider anisotropic $SU(2)$ spin models with Hamiltonians of the form $H = \sum_{\langle i,j\rangle}[J^3(i)J^3(j) + K(h(i), h(j))]$, where $h(i) \equiv a_1^\dagger(i)a_2^\dagger(i)$ and K is a Hermitian operator depending on $h(i)$ and $h(j)$. This Hamiltonian does not commute with $\vec{J}(i)\cdot\vec{J}(i)$ because of the presence of $a_1^\dagger a_2^\dagger$ and a_1a_2 terms. These terms change the corresponding spin value by ± 1, respectively, as is clear from (8). However, these terms do not change the value of $J^3 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right)$. Therefore, to study such Hamiltonians, the fixed charge (J^3) coherent states in (18) or (24) should be useful.

We note that the construction of $SU(N)(N \geqslant 3)$ charge coherent states involved only symmetric representations of $SU(N)$. This was the reason we were led to a simple $SU(N)$ generalization of the $SU(2)$ results. It would be interesting to consider all the irreducible representations of $SU(N)$ and define the most general $SU(N)$ charge coherent states. This can be done by including $(N-1)$ sets of harmonic oscillators belonging to all the $(N-1)$ fundamental representations of $SU(N)$ [7]. Work in this direction is in progress and will be reported elsewhere.

Appendix

In this appendix, we prove that the $SU(3)$ charge and hyper-charge coherent states in (40) satisfy the resolution of identity property

$$\mathcal{I}_{q,l} = \int \mathrm{d}^2z_1\,\mathrm{d}^2z_2\,\mathrm{d}^2z_3\delta(|z_1|^2 + |z_2|^2 + |z_3|^2 - 1)|z_1, z_2, z_3\rangle_{q,l\,q,l}\langle z_1, z_2, z_3|. \tag{A.1}$$

To solve the δ function constraints, it is convenient to define

$$z_1 = r_1\,\mathrm{e}^{\mathrm{i}\theta_1}, \qquad z_2 = r_2\,\mathrm{e}^{\mathrm{i}\theta_2}, \qquad z_3 = r_3\,\mathrm{e}^{\mathrm{i}\theta_3}.$$

After integrating out the angles θ_1, θ_2 and θ_3, we get

$$\mathcal{I}_{q,l} = C\sum_{n=0}^{\infty}\frac{(3n+2l-q+2)!}{(n+l)!(n+l-q)!n!}\int r_1\,\mathrm{d}r_1 r_2\,\mathrm{d}r_2 r_3\,\mathrm{d}r_3\delta\left(r_1^2 + r_2^2 + r_3^2 - 1\right)\left(r_1^2\right)^{n+l}\left(r_2^2\right)^{n+l-q}$$
$$\times\left(r_3^2\right)^n|n+l, n+l-q, n\rangle\langle n+l, n+l-q, n|. \tag{A.2}$$

In (A.2), C is a constant. In terms of the above radial coordinates, the δ function constraint can be solved by polar decomposition, i.e.,

$$r_1 = r\sin\theta\cos\phi, \qquad r_2 = r\sin\theta\sin\phi, \qquad r_3 = r\cos\theta. \tag{A.3}$$

In (A.3), since $r_1, r_2, r_3 \geqslant 0$, both θ and ϕ vary between 0 and $\frac{\pi}{2}$. Now r can be trivially integrated off because of the δ function and we are left with

$$\mathcal{I}_{q,l} = C\sum_{n=0}^{\infty}\frac{(3n+2l-q+2)!}{(n+l)!(n+l-q)!n!}\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\,\mathrm{d}\theta(\sin^2\theta)^{2n+2l-q+1}(\cos^2\theta)^n$$
$$\times\int_0^{\frac{\pi}{2}}\sin\phi\cos\phi\,\mathrm{d}\phi(\sin^2\phi)^{n+l-q}(\cos^2\phi)^{n+l}|n+l, n+l-q, n\rangle\langle n+l, n+l-q, n|. \tag{A.4}$$

We now set $x = \sin^2\theta$, $y = \sin^2\phi$ to get

$$\mathcal{I}_{q,l} = C\sum_{n=0}^{\infty}\frac{(3n+2l-q+2)!}{(n+l)!(n+l-q)!n!}\int_0^1\mathrm{d}x\,x^{2n+2l-q+1}(1-x)^n\int_0^1\mathrm{d}y\,y^{n+l-q}(1-y)^{n+l}$$
$$\times|n+l, n+l-q, n\rangle\langle n+l, n+l-q, n|. \tag{A.5}$$

Using the fact

$$B(m,n) = \int_0^1\mathrm{d}x\,x^{m-1}(1-x)^{n-1} = \frac{(m-1)!(n-1)!}{(m+n-1)!},$$

we get

$$\mathcal{I}_{q,l} = C\sum_{n=0}^{\infty}|n+l, n+l-q, n\rangle\langle n+l, n+l-q, n|.$$

Therefore, $\mathcal{I}_{q,l}$ is an identity operator in the Fock space of three harmonic oscillators with fixed $SU(3)$ charge and hyper-charge. In the full Hilbert space of three harmonic oscillators, $\mathcal{I}_{q,l}$ is the projection operator which projects out the fixed charge component characterized by q and l. The above proof of resolution of identity can be easily generalized to $SU(N)$ by using polar decomposition analogous to (A.3) on S^{2N-1}.

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